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NASA TN D-1177

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TECHNICAL NOTE

D-1177

COMPUTATIONAL PROCEDURE FOR VINTI'S THEORY OF AN ACCURATE INTERMEDIARY ORBIT

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON March 1962

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SUMMARY

By introducing the oblate spheroidal system of generalized coordinates into the solution of Laplace's equation, three adjustable constants are provided by which this solution can be made to agree largely with the earth's potential expressed by means of a general expansion in spherical harmonics. This agreement is exact for the zeroth, first, and second zonal harmonics, and as a consequence of this system, through more than half of the latest accepted value of the earth's fourth harmonic. Based on this theory of Vinti's solution by separable Hamiltonian, a computing procedure is described for obtaining the coordinates and velocity of an unretarded satellite from a knowledge of its initial conditions.

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INTRODUCTION

Vinti (Reference 1) has found a gravitational potential for an axially symmetric planet in oblate spheroidal coordinates which simultaneously satisfies Laplace's equation and separates the Hamilton-Jacobi equation. This potential is given by the expression

$$V(\rho, \eta) = \frac{-\mu\rho}{\rho^2 + c^2\eta^2} .$$

It accounts for all of the second zonal harmonic and for more than half of the fourth zonal harmonic. This solution has been shown to fit the experimentally observed values of the earth's gravitational potential within ± 0.2 parts per million. This succeeds in reducing the problem of satellite motion to quadratures, with the use of a potential that is much closer to the empirically accepted one for the earth than any previously used as the starting point of a calculation. If we consider a 2n dimensional cartesian space formed of coordinates $q_1 \dots q_n$, $p_1 \dots p_n$, and known as phase space, the complete dynamic specification of a mechanical system will be given by a point in such a space. The values of q and p at any time t can be obtained from their initial values by a canonical transformation which is a continuous function of time, that is, there exists a canonical transformation from the values of the coordinates and momenta at any time t to their initial values. Obtaining this transformation is equivalent to solving the problem of the system motion. Accordingly, the motion of a mechanical system corresponds to the continuous evolution or unfolding of a canonical transformation.

Vinti's solution by means of the canonical transformation for which both the momenta and coordinates are constants of the motion α_1 and β_1 differs from others in that the oblateness potential is included in the solution of the equations of motion. By use of one half of the transformation equations relating the α's with the initial q and p values at the initial time, Vinti also evaluates these constants in terms of specific initial conditions of the problem.

A procedure for obtaining the constant β's from initial conditions in Vinti's kinetic equations, representing the other half of the equations of transformation is included. Elliptic integrals appearing in these latter equations have been replaced by rapidly converging infinite series, and the inversion of the resulting kinetic equations then furnishes the coordinates q as functions of the initial conditions α_1, β_1, and the time.

A method for obtaining the velocity components from this theory is also described.

STATEMENT OF THE PROBLEM

If we take the latest values given by W. M. Kaula $\left[\mu = GM = 398.6032 \pm 0.0032\right.$ $mm^3/ksec^2$, where G is the gravitational constant and M is the mass of the earth, $A_E = 6.378165\ (\pm\ 0.000025)\,mm$, equal to the equatorial radius of the earth, and $J_2 = 1.08230\ (\pm\ 0.2) \times 10^{-3}$, equal to the coefficient of the second harmonic$\left.\right]$, then from the relation $C^2 = J_2 A_E^2$ we have $C = 0.209831$ mm.

The relation between the oblate spheroidal and the geocentric rectangular coordinate systems is given by the following equations from Reference 2:

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)}\cos\phi, \qquad (0 < \rho < \infty)\ ;$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)}\sin\phi, \qquad (-1 \leqq \eta \leqq 1)\ ;$$

$$Z = \rho\eta;$$

where ρ, η, and ϕ are the coordinates in the oblate spheroidal system.

The Lagrangian is defined as

$$L = \frac{1}{2}\left(\frac{dS}{dt}\right)^2 + \frac{\mu\rho}{\rho^2 + c^2\eta^2}$$

where the curvilinear velocity is

$$\left(\frac{dS}{dt}\right)^2 = h_\rho^2\,\dot{\rho}^2 + h_\eta^2\,\dot{\eta}^2 + h_\phi^2\,\dot{\phi}^2\ ,$$

the squares of the scale factors are

$$h_\rho^2 = \frac{\rho^2 + c^2\eta^2}{\rho^2 + c^2},$$

$$h_\eta^2 = \frac{\rho^2 + c^2\eta^2}{1 - \eta^2},$$

$$h_\phi^2 = \left(\rho^2 + c^2\right)\left(1 - \eta^2\right),$$

and the generalized momenta are obtained from

$$P_\rho = \frac{\partial L}{\partial \dot{\rho}},$$

$$P_\eta = \frac{\partial L}{\partial \dot{\eta}},$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}}.$$

It can be shown that if L is not an explicit function of time, then the Hamiltonian H is a constant of the motion. Furthermore, since the equations of transformation to the (generalized) oblate coordinates do not depend explicitly on the time, and since the potential is taken as velocity independent, then H is the total energy and is given by

$$H = h_\rho^{-2}\left(\frac{\partial W}{\partial \rho}\right)^2 + h_\eta^{-2}\left(\frac{\partial W}{\partial \eta}\right)^2 + h_\phi^{-2}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{\mu\rho}{\rho^2 + c^2\eta^2} = \alpha_1,$$

where W is Hamilton's characteristic function.

Using this, Vinti's dynamical equations of motion are seen to be (Reference 1)

$$t + \beta_1 = \pm\int_{\rho_1}^{\rho} \rho^2 F(\rho)^{-\frac{1}{2}}\, d\rho \pm c^2\int_0^{\eta} \eta^2 G(\eta)^{-\frac{1}{2}}\, d\eta,$$

$$\beta_2 = \mp\alpha_2\int_{\rho_1}^{\rho} F(\rho)^{-\frac{1}{2}}\, d\rho \pm \alpha_2\int_0^{\eta} G(\eta)^{-\frac{1}{2}}\, d\eta,$$

$$\phi - \beta_3 = \mp c^2\alpha_3\int_{\rho_1}^{\rho}\left(\rho^2 + c^2\right)^{-1} F(\rho)^{-\frac{1}{2}} d\rho \pm \alpha_3\int_0^{\eta}\left(1 - \eta^2\right)^{-1} G(\eta)^{-\frac{1}{2}}\, d\eta,$$

where $F(\rho)$ and $G(\eta)$ are the quartic polynomials and

$$F(\rho) = c^2\alpha_3^2 + \left(\rho^2 + c^2\right)\left(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2\right),$$

$$G(\eta) = -\alpha_3^2 + \left(1 - \eta^2\right)\left(\alpha_2^2 + 2\alpha_1 c^2\eta^2\right).$$

The α's and β's are the Jacobi constants, with energy $\alpha_1 < 0$.

In the limiting case $(c \rightarrow 0)$ of Keplerian motion, we have:

α_1, the total energy in the orbit;
α_2, the total angular momentum;
α_3, the polar component of angular momentum;
β_1, the time of perigee passage;
β_2, the argument of perigee;
β_3, the right ascension of the ascending node.

According to Reference 3, the values of α_1, α_2, and α_3 are determined rigorously from the initial coordinates and their derivatives, and are given by

$$\alpha_1 = \frac{1}{2} U_i^2 - \mu\rho_i\left(\rho_i^2 + c^2\eta_i^2\right)^{-1} ,$$

$$\alpha_3 = X_i\dot{Y}_i - Y_i\dot{X}_i ,$$

$$\alpha_2 = \left(1 - \eta_i^2\right)^{-1} \left[\left(\rho_i^2 + c^2\eta_i^2\right)^2\dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2\eta_i^2\left(1 - \eta_i^2\right)\right] .$$

Here U_i denotes the speed.

A knowledge of these quantities permits a numerical solution of the quartic equation $F(\rho) = 0$ and furnishes the numerical values of ρ_1, ρ_2, A, and B necessary to factor $F(\rho)$ into the form

$$F(\rho) = -2\alpha_1\left(\rho - \rho_1\right)\left(\rho_2 - \rho\right)\left(\rho^2 - A\rho + B\right) ,$$

where $\rho_1 = a(1 - e)$ and $\rho_2 = a(1 + e)$ are zeros of $F(\rho)$. The quantities A, B, a, and e are given by Equations (3.19), (3.20), (3.23), and (3.24) of Reference 2.

Once mean values of the Jacobi constants α_1, α_2, α_3, β_1, β_2, and β_3 are obtained over many revolutions of an orbit, it then becomes more advantageous to compute A and B by Equations (4.12) and (4.13) of Reference 3 instead of (3.19) and (3.20), since the Jacobi constants can now help factor the two quartics $F(\rho)$ and $G(\eta)$ exactly.

THE VELOCITIES $\dot{X}$, $\dot{Y}$, AND $\dot{Z}$

If we are given the initial conditions as X_i, Y_i, Z_i, $\dot{X}_i$, $\dot{Y}_i$, and $\dot{Z}_i$ for an initial time t_i, we can write

$$r_i^2 = X_i^2 + Y_i^2 + Z_i^2 ,$$

$$r_i\dot{r}_i = X_i\dot{X}_i + Y_i\dot{Y}_i + Z_i\dot{Z}_i .$$

Transforming to the oblate coordinates, we obtain

$$\rho_i = \left[\frac{r_i^2 - c^2}{2} \left(1 + \sqrt{1 + \frac{4Z_i^2 c^2}{\left(r_i^2 - c^2\right)^2}} \right) \right]^{\frac{1}{2}} ,$$

$$\eta_i = \left[\frac{2Z_i^2}{r_i^2 - c^2} \left(1 + \sqrt{1 + \left(\frac{2cZ_i}{r_i^2 - c^2} \right)^2} \right)^{-1} \right]^{\frac{1}{2}} ,$$

where the sign of η_i is taken to be the same as that of Z_1. By differentiation, the velocities are found to be

$$\dot{\eta}_i = \left(\frac{1}{2\eta_i c^2} \right) \left[- r_i \dot{r}_i + \frac{2c^2 Z_i \dot{Z}_i + \left(r_i^2 - c^2\right) r_i \dot{r}_i}{\sqrt{\left(c^2 - r_i^2\right)^2 + 4c^2 Z_i^2}} \right],$$

$$\dot{\rho}_i = \frac{r_i \dot{r}_i}{2\rho_i} + \frac{2c^2 Z_i \dot{Z}_i - \left(c^2 - r_i^2\right) r_i \dot{r}_i}{2\rho_i \sqrt{\left(c^2 - r_i^2\right)^2 + 4c^2 Z_i^2}} .$$

To define the uniformizing variables E, v, and ψ given by Vinti as the "eccentric anomaly", the "true anomaly", and a variable analogous to the argument of latitude, respectively, at the initial time, the following procedure is employed:

From Reference 1 we have

$$\dot{\rho} = \frac{P_\rho}{h_\rho^2} = \pm \frac{\sqrt{F(\rho)}}{h_\rho^2 \left(\rho^2 + c^2\right)} ,$$

$$\dot{\eta} = \frac{P_\eta}{h_\eta^2} = \pm \frac{\sqrt{G(\eta)}}{h_\eta^2 \left(1 - \eta^2\right)} ,$$

$$\dot{\phi} = \frac{P_\phi}{h_\phi^2} = \pm \frac{\alpha_3}{h_\phi^2} ,$$

with $\alpha_3 \gtrless 0$ as the orbit is direct or retrograde.

Substituting $\rho_1 = a(1 - e)$, $\rho_2 = a(1 + e)$, $\rho = a(1 - e \cos E)$, and $\eta = \eta_0 \sin \psi$ into

$$F(\rho) = -2\alpha_1 \left(\rho - \rho_1\right) \left(\rho_2 - \rho\right) \left(\rho^2 + A\rho + B\right)$$

and

$$G(\eta) = -2\alpha_1 c^2 (\eta_0^2 - \eta^2)(\eta_2^2 - \eta^2) ,$$

and replacing in the above expressions for $\sqrt{F(\rho)}$ and $\sqrt{G(\eta)}$, we can eliminate the $\pm$ signs in favor of the uniformizing variables, since now, at $t = t_i$,

$$\dot{\rho}_i = \frac{ae\sqrt{-2\alpha_1(\rho_i^2 + A\rho_i + B)}}{h_{\rho_i}^2\left(\rho_i^2 + c^2\right)} \sin E_i ,$$

$$\cos E_i = \left(\frac{1 - \rho_i}{a}\right) e^{-1} ,$$

$$\dot{\eta}_i = \frac{c\eta_0\sqrt{-2\alpha_1(\eta_2^2 - \eta_i^2)}}{h_{\eta_i}^2\left(1 - \eta_i^2\right)} \cos \psi_i ,$$

and

$$\sin \psi_i = \frac{\eta_i}{\eta_0}$$

completely determine ψ_i and E_i, and v_i is determined from the anomally connections

$$\cos v_i = \frac{\cos E_i - e}{1 - e \cos E_i} ,$$

and

$$\sin v_i = \frac{(1 - e^2)^{\frac{1}{2}} \sin E_i}{1 - e \cos E_i} .$$

When $e = 0$, then $v_i = E_i = 0$.

If we now differentiate

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \cos \phi ,$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \sin \phi ,$$

and

$$Z = \rho\eta$$

with respect to time and substitute in the above results for $\dot{\rho}$, $\dot{\eta}$, and $\dot{\phi}$ for any time t, we have

$$\dot{X} = X\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) - \frac{Y\alpha_3}{h_\phi^2} ,$$

$$\dot{Y} = Y\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) + \frac{X\alpha_3}{h_\phi^2} ,$$

$$\dot{Z} = \rho\dot{\eta} + \eta\dot{\rho} .$$

The Jacobi-Constants β_1, β_2 and β_3

In addition to the uniformizing variables E_i, v_i, and ψ_i, the geocentric right ascension ϕ_i and the angle χ_i from Equation (6.51) of Reference 3 can be determined at the initial time by the expressions

$$\sin \phi_i = \frac{Y_i}{\sqrt{\rho_i^2 + c^2}\sqrt{1 - \eta_i^2}} ,$$

$$\cos \phi_i = \frac{X_i}{\sqrt{\rho_i^2 + c^2}\sqrt{1 - \eta_i^2}} ,$$

$$\sin \chi_i = \frac{\sqrt{1 - \eta_0^2}\ \sin \psi_i}{\sqrt{1 - \eta_0{}^2 \sin^2 \psi_i}} ,$$

and

$$\cos \chi_i = \frac{\cos \psi_i}{\sqrt{1 - \eta_0^2 \sin^2 \psi_i}} .$$

Assembling and substituting the above results in Vinti's kinetic equations [Equations (8.2), (8.3), and (8.50) of Reference 3], the Jacobi constants are then solved for and are given as

$$\beta_1 = \left(-2\alpha_1\right)^{-\frac{1}{2}} \left[b_i E_i + a\left(E_i - e \sin E_i\right) + A_i v_i + A_{11} \sin v_i + A_{12} \sin 2 v_i\right] + c^2\left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0^3 \left[B_1 \psi_i - \frac{1}{8}\left(2 + q^2\right) \sin 2 \psi_i + \frac{q^2}{64} \sin 4 \psi_i\right] - t_i ,$$

$$\beta_2 = -\alpha_2\left(-2\alpha_1\right)^{-\frac{1}{2}} \left[A_2 v_i + A_{21} \sin v_i + A_{22} \sin 2 v_i + A_{23} \sin 3 v_i + A_{24} \sin 4 v_i\right] + \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0 \alpha_2 \left[B_2 \psi_i - \frac{q^2}{32}\left(4 + 3q^2\right) \sin 2 \psi_i + \left(\frac{3q^4}{256}\right) \sin 4 \psi_i\right],$$

and

$$\beta_3 = \phi_i - \alpha_3\left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0 \left[\left(1 - \eta_0^2\right)^{-\frac{1}{2}} \left(1 - \eta_2^{-2}\right)^{-\frac{1}{2}} \chi_i + B_3 \psi_i + \frac{3}{32} \eta_0^2 \eta_2^{-4} \sin 2 \psi_i\right] + c^2 \alpha_3 \left(-2\alpha_1\right)^{-\frac{1}{2}} \left[A_3 v_i + \sum_{n=1}^{4} A_{3n} \sin n v_i\right].$$

COMPUTATIONAL PROCEDURE

All symbols used in the following paragraphs are consistent with References 1 and 3.

The following paragraphs give the steps of the computational procedure that is used to produce motion of the satellite.

Coordinate Conversion and Jacobi Constants (Generalized Momenta)

1. Enter the constants μ and c.
2. Enter the initial conditions (injection vectors) X_i, Y_i, Z_i, $\dot{X}_i$, $\dot{Y}_i$ and $\dot{Z}_i$ with t_i.
3. Compute:

$$r_i = \sqrt{X_i^2 + Y_i^2 + Z_i^2} ,$$

$$r_i \dot{r}_i = X_i \dot{X}_i + Y_i \dot{Y}_i + Z_i \dot{Z}_i ,$$

$$\rho_i^2 = \frac{r_i^2 - c^2}{2}\left(1 + \sqrt{1 + \frac{4Z_i^2 c^2}{\left(r_i^2 - c^2\right)^2}}\right) ,$$

$$\rho_i = \left[\frac{r_i^2 - c^2}{2}\left(1 + \sqrt{1 + \frac{4Z_i^2 c^2}{\left(r_i^2 - c^2\right)^2}}\right)\right]^{\frac{1}{2}} ,$$

$$\eta_i^2 = \frac{2Z_i^2}{r_i^2 - c^2}\left[1 + \sqrt{1 + \left(\frac{2cZ_i}{r_i^2 - c^2}\right)^2}\,\right]^{-1} ,$$

$\eta_i = \sqrt{\eta_i^2}$, where the sign of η_i = sign of Z_i,

$$\dot{\eta}_i = \left(\frac{1}{2\eta_i c^2}\right)\left[-r_i\dot{r}_i + \frac{2c^2 Z_i \dot{Z}_i + \left(r_i^2 - c^2\right) r_i \dot{r}_i}{\sqrt{\left(c^2 - r_i^2\right)^2 + 4c^2 Z_i^2}}\right] ,$$

$\dot{\eta}_i^2$,

$$\dot{\rho}_i = \frac{r_i \dot{r}_i}{2\rho_i} + \frac{2c^2 Z_i \dot{Z}_i - \left(c^2 - r_i^2\right) r_i \dot{r}_i}{2\rho_i \sqrt{\left(c^2 - r_i^2\right)^2 + 4c^2 Z_i^2}} ,$$

$$\cos \phi_i = \frac{X_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1 - \eta_i^2}} ,$$

$$\sin \phi_i = \frac{Y_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1 - \eta_i^2}} ,$$

then obtain ϕ_i within the limits $0 \le \phi_i < 2\pi$,

$$\alpha_1 = \frac{\dot{X}_i^2 + \dot{Y}_i^2 + \dot{Z}_i^2}{2} - \mu\rho_i \left(\rho_i^2 + c^2\eta_i^2\right)^{-1} ,$$

$$\alpha_3 = X_i\dot{Y}_i - Y_i\dot{X}_i$$

$$\alpha_3^2 ,$$

$$\alpha_2{}^2 = \left(1 - \eta_i^2\right)^{-1} \left[\left(\rho_i^2 + c^2\eta_i^2\right)^2 \dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2\eta_i^2 \left(1 - \eta_i^2\right) \right] .$$

Prime Constants

Compute:

$$X_D^2 = -2\alpha_1 \alpha_2^2 \mu^{-2} ,$$

$$X_D^4 ,$$

$$p_0 = \frac{\alpha_2^2}{\mu} ,$$

$$p_0^2 ,$$

$$Y_D^2 = \left(\cos^2 i_0\right) = \left(\frac{\alpha_3}{\alpha_2}\right)^2 ,$$

$$Y_D^4 ,$$

$$\sin i_0 = \sqrt{1 - Y_D^2} ,$$

$$\sin^2 i_0 ,$$

$$K_0 = \frac{c^2}{p_0^2} ,$$

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K_0^2 ,

$$A = -2K_0 p_0 Y_D^2 \left[1 + K_0 \left(2X_D^2 - 3X_D^2 Y_D^2 - 4 + 8Y_D^2\right) \right] ,$$

$$B = K_0 p_0^2 \left(1 - Y_D^2\right) \left[1 + K_0 \left(4Y_D^2 - X_D^2 Y_D^2\right) \right] ,$$

$$b_1 = -\frac{1}{2} A ,$$

$$b_2 = B^{\frac{1}{2}} ,$$

$$\left(\rho_1 + \rho_2\right) = 2a = 2p_0 X_D^{-2} \left[1 - K_0 X_D^2 Y_D^2 - K_0^2 X_D^2 Y_D^2 \left(2X_D^2 - 3X_D^2 Y_D^2 - 4 + 8Y_D^2\right) \right] ,$$

$$\rho_1 \rho_2 = ap = p_0^2 X_D^{-2} \left[1 + K_0 Y_D^2 \left(X_D^2 - 4\right) - K_0^2 Y_D^2 \left(12X_D^2 - X_D^4 - 20X_D^2 Y_D^2 - 16 + 32Y_D^2 + X_D^4 Y_D^2\right) \right] ,$$

$$a = \left(\frac{\rho_1 + \rho_2}{2}\right),$$

$$g = \frac{4\rho_1 \rho_2}{\left(\rho_1 + \rho_2\right)^2} ,$$

$$p = \frac{2\rho_1 \rho_2}{\left(\rho_1 + \rho_2\right)} ,$$

$$e = \sqrt{1 - g} ,$$

$$\begin{pmatrix} \eta_0^{-2} \\ \eta_2^{-2} \end{pmatrix} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2\left(\alpha_2^2 - \alpha_3^2\right)} \left\{ 1 \pm \left[1 + \frac{8\alpha_1 c^2 \left(\alpha_2^2 - \alpha_3^2\right)}{\left(\alpha_2^2 - 2\alpha_1 c^2\right)^2} \right]^{\frac{1}{2}} \right\},$$

$$q^2 = \left(\frac{\eta_0}{\eta_2}\right)^2,$$

and

$$q^4.$$

The terms η_i and i_0 are now tested to insure that they fall within the following limits

$$-1 \leqq -\eta_0 \leqq \eta_i \leqq \eta_0 \leqq 1$$

and

$$I_c \leqq i_0 \leqq 180° - I_c ,$$

where $I_c = 1°54'$.

NOTE: If α's and β's are corrected by observations, then Equations (3.12) through (3.15) of Reference 3 can be solved for $\left(\rho_1 + \rho_2\right)$, $\rho_1\rho_2$, A, and B, to arbitrarily high order by a Newton-Raphson iterative scheme.

Mutual Constants

1. Compute:

$$\left(1 - e^2\right)^{\frac{1}{2}},$$

$$\left(\frac{b_1}{b_2}\right),$$

$$A_1 = \left(1 - e^2\right)^{\frac{1}{2}} p \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[\left(1 - e^2\right)^{\frac{1}{2}}\right] ,$$

$$A_2 = \left(1 - e^2\right)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_n\left[\left(1 - e^2\right)^{\frac{1}{2}}\right] ,$$

where $P_n\left(\frac{b_1}{b_2}\right)$ is the Legendre Polynomial of degree n, $R_n(X_s) = X_s^n P_n\left(X_s^{-1}\right)$ is a polynomial of degree $\left(\frac{n}{2}\right)$ in X_s^2, and $X_s = \left(1 - e^2\right)^{\frac{1}{2}}$.

2. <u>Test</u>

If m is an even integer, compute

$$D_m = D_{2i} = \sum_{n=0}^{i} (-1)^{i-n}\left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} P_{2n}\left(\frac{b_1}{b_2}\right) .$$

If m is an odd integer, compute

$$D_m = D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n}\left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} P_{2n+1}\left(\frac{b_1}{b_2}\right) .$$

3. Then compute:

$$A_3 = \left(1 - e^2\right)^{\frac{1}{2}} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2}\left[\left(1 - e^2\right)^{\frac{1}{2}}\right] ,$$

$$B_1 = \frac{1}{2} + \frac{3}{16}q^2 + \frac{15}{128} q^4 ,$$

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$$B_2 = 1 + \frac{q^2}{4} + \frac{9}{64} q^4 ,$$

$$\gamma_m = \frac{(2m)!}{2^{2m} (m!)^2} \sum_{n=1}^{m-1} \frac{(2n)! \ \eta_0^{2n}}{2^{2n} (n!)^2} ,$$

$$B_3 = 1 - \left(1 - \eta_2^{-2}\right)^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m} ,$$

$$A_{11} = \frac{3}{4}\left(1 - e^2\right)^{\frac{1}{2}} p^{-3} e \left(- 2b_1 b_2^2 \ p + b_2^4\right) ,$$

$$A_{12} = \frac{3}{32}\left(1 - e^2\right)^{\frac{1}{2}} b_2^4 \ e^2 \ p^{-3} ,$$

$$A_{21} = \left(1 - e^2\right)^{\frac{1}{2}} p^{-1} e \left[b_1 p^{-1} + \left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1 + \frac{e^2}{4}\right) p^{-3} + \frac{3}{8} b_2^4 \left(4 + 3e^2\right) p^{-4} \right] ,$$

$$A_{22} = \left(1 - e^2\right)^{\frac{1}{2}} p^{-1} \left[\frac{e^2}{8}\left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 \left(6e^2 + e^4\right) p^{-4}\right] ,$$

$$A_{23} = \left(1 - e^2\right)^{\frac{1}{2}} p^{-1} \frac{e^3}{8} \left(- b_1 b_2^2 \ p^{-3} + b_2^4 \ p^{-4}\right) ,$$

$$A_{24} = \frac{3}{256}\left(1 - e^2\right)^{\frac{1}{2}} p^{-5} \ b_2^4 \ e^4 ,$$

$$A_{31} = \left(1 - e^2\right)^{\frac{1}{2}} p^{-3}\, e\left[2 + b_1\, p^{-1}\left(3 + \frac{3}{4}\, e^2\right) - p^{-2}\left(\frac{1}{2}\, b_2^{\,2} + c^2\right)\left(4 + 3e^2\right)\right] ,$$

$$A_{32} = \left(1 - e^2\right)^{\frac{1}{2}} p^{-3}\left[\frac{e^2}{4} + \frac{3}{4}\, b_1 p^{-1}\, e^2 - p^{-2}\left(\frac{b_2^{\,2}}{2} + c^2\right)\left(\frac{3}{2}e^2 + \frac{e^4}{4}\right)\right],$$

$$A_{33} = \left(1 - e^2\right)^{\frac{1}{2}} e^3\left[\frac{b_1}{12p} - \frac{p^{-2}}{3}\left(\frac{b_2^{\,2}}{2} + c^2\right)\right] p^{-3} ,$$

$$A_{34} = -\frac{1}{32}\left(1 - e^2\right)^{\frac{1}{2}} p^{-5}\, e^4\left(\frac{1}{2}\, b_2^{\,2} + c^2\right) ,$$

$$2\pi\gamma_1 = \left(-2\alpha_1\right)^{\frac{1}{2}}\left(a + b_1 + A_1 + c^2\eta_0^2\, A_2 B_1 B_2^{\,-1}\right)^{-1} ,$$

$$2\pi\gamma_2 = \left(\alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}} \eta_0^{-1} A_2\, B_2^{\,-1}\left(a + b_1 + A_1 + c^2\eta_0^2\, A_2 B_1 B_2^{\,-1}\right)^{-1} ,$$

$$e' = ae\left(a + b_1\right)^{-1} ,$$

the parameter e' is always less than e.

Jacobi-Constants (Generalized Coordinates)

If $e = 0$, then $v_i = E_i = 0$.

If $e \neq 0$, then compute:

$$h_{\rho_i}^2 = \frac{\rho_i^2 + \eta_i^2\, c^2}{\rho_i^2 + c^2} ,$$

$$h_{\eta_i}^2 = \frac{\rho_i^2 + \eta_i^2 c^2}{1 - \eta_i^2} ,$$

$$h_{\phi_i}^2 = \left(\rho_i^2 + c^2\right)\left(1 - \eta_i^2\right) ,$$

$$\sin E_i = \frac{\dot{\rho}_i \, h_{\rho_i}^2 \left(\rho_i^2 + c^2\right)}{ae\sqrt{-2\alpha_1\left(\rho_i^2 + A\rho_i + B\right)}} ,$$

$$\cos E_i = \left(1 - \frac{\rho_i}{a}\right)e^{-1} ,$$

now obtain E_i within the limits $0 \leq E_i < 2\pi$,

$$\cos \psi_i = \frac{\dot{\eta}_i \, h_{\eta_i}^2 \left(1 - \eta_i^2\right)}{c\eta_0 \sqrt{-2\alpha_1\left(\eta_2^2 - \eta_i^2\right)}} ,$$

$$\sin \psi_i = \frac{\eta_i}{\eta_0} ,$$

determine ψ_i within the limits $0 \leq \psi_i < 2\pi$,

$$\cos v_i = \frac{\cos E_i - e}{1 - e \cos E_i} ,$$

$$\sin v_i = \frac{\left(1 - e^2\right)^{\frac{1}{2}} \sin E_i}{1 - e \cos E_i} ,$$

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determine v_i within the limits $0 \leq v_i < 2\pi$,

$\sin n\, v_i$ for $n = 2, 3, 4,$

$\sin n\, \psi_i$ for $n = 2, 4,$

$$\sin \chi_i = \frac{\sqrt{1 - \eta_0^2}\ \sin \psi_i}{\sqrt{1 - \eta_0^2 \sin^2 \psi_i}} ,$$

$$\cos \chi_i = \frac{\cos \psi_i}{\sqrt{1 - \eta_0^2 \sin^2 \psi_i}} ,$$

determine χ_i within the limits $0 \leq \chi_i < 2\pi$,

$$\beta_1 = \left(-2\alpha_1\right)^{-\frac{1}{2}} \left[b_1 E_i + a\left(E_i - e \sin E_i\right) + A_1 v_i + A_{11} \sin v_i + A_{12} \sin 2\, v_i \right]$$
$$+ c^2 \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0^3 \left[B_1\, \psi_i - \frac{1}{8}\left(2 + q^2\right) \sin 2\, \psi_i + \frac{q^2}{64} \sin 4\, \psi_i \right] - t_i ,$$

$$\beta_2 = -\alpha_2 \left(-2\alpha_1\right)^{-\frac{1}{2}} \left[A_2\, v_i + A_{21} \sin v_i + A_{22} \sin 2\, v_i + A_{23} \sin 3\, v_i + A_{24} \sin 4\, v_i \right]$$
$$+ \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0\, \alpha_2 \left[B_2\, \psi_i - \frac{q^2}{32}\left(4 + 3q^2\right) \sin 2\, \psi_i + \frac{3q^4}{256} \sin 4\, \psi_i \right] ,$$

$$\beta_3 = \phi_i - \alpha_3 \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0 \left[\left(1 - \eta_0^2\right)^{-\frac{1}{2}} \left(1 - \eta_2^{-2}\right)^{-\frac{1}{2}} \chi_i + B_3\, \psi_i \right.$$
$$\left. + \frac{3}{32} \eta_0^2\, \eta_2^{-4} \sin 2\, \psi_i \right] + c^2 \alpha_3 \left(-2\alpha_1\right)^{-\frac{1}{2}} \left[A_3\, v_i + \sum_{n=1}^{4} A_{3n} \sin n\, v_i \right] .$$

Orbit Generator

Compute for any time t:

$$M_s = 2\pi\,\gamma_1 \left[t + \beta_1 - c^2\,\beta_2\,\alpha_2^{-1}\,\eta_0^2\,B_1 B_2^{-1} \right] ,$$

$$\psi_s = 2\pi\,\gamma_2 \left[t + \beta_1 + \beta_2\,\alpha_2^{-1} \left(a + b_1 + A_1 \right) A_2^{-1} \right] ,$$

Also, determine M_s and ψ_s within 2π,
By the Newton-Raphson iteration:

$$\mathcal{E} = \mathcal{E}_{n+1} = \mathcal{E}_n - \frac{[\mathcal{E}_n - e' \sin \mathcal{E}_n - M_s]}{(1 - e' \cos \mathcal{E}_n)}$$

$$- \frac{1}{2} \left[\frac{\mathcal{E}_n - e' \sin \mathcal{E}_n - M_s}{1 - e' \cos \mathcal{E}_n} \right]^2 \left[\frac{e' \sin \mathcal{E}_n}{1 - e' \cos \mathcal{E}_n} \right],$$

where $\mathcal{E}_n = M_s$ initially,

Note, all subsequent secular terms of the generator are non-modulo,

$$\cos v' = \left(\cos \mathcal{E} - e \right) \left(1 - e \cos \mathcal{E} \right)^{-1} ,$$

$$\sin v' = \left(1 - e^2 \right)^{\frac{1}{2}} \left(1 - e \cos \mathcal{E} \right)^{-1} \sin \mathcal{E} ,$$

also, determine v' within the limits $0 \le v' < 2\pi$,

$$v_0 = v' - M_s ,$$

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$$\psi_0 = \left(-2\alpha_1\right)^{-\frac{1}{2}} \left(\alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}} \eta_0^{-1} A_2 B_2^{-1} v_0 ,$$

$$M_1 = \left(a + b_1\right)^{-1} \left[- \left(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right) v_0 \right.$$

$$\left. + \frac{c^2}{4} \left(-2\alpha_1\right)^{\frac{1}{2}} \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0^3 \sin\left(2\psi_s + 2\psi_0\right) \right] ,$$

$$E_1 = \left[1 - e' \cos \mathcal{E} \right]^{-1} M_1 - \frac{e'}{2} \left[1 - e' \cos \mathcal{E} \right]^{-3} M_1^2 \sin \mathcal{E} ,$$

$$\cos v'' = \left[\cos\left(\mathcal{E} + E_1\right) - e \right] \left[1 - e \cos\left(\mathcal{E} + E_1\right) \right]^{-1} ,$$

$$\sin v'' = \left(1 - e^2\right)^{\frac{1}{2}} \left[1 - e \cos\left(\mathcal{E} + E_1\right) \right]^{-1} \sin\left(\mathcal{E} + E_1\right) ,$$

also, determine v'' within the limits $0 \le v'' < 2\pi$,

$$v_1 = v'' - \left(v_0 + M_s\right) = \left(v'' - v'\right) ,$$

$$\psi_1 = \left(-2\alpha_1\right)^{-\frac{1}{2}} \left(\alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}} \eta_0^{-1} B_2^{-1} \left[A_2 v_1 + A_{21} \sin v' + A_{22} \sin 2v' \right]$$

$$+ \frac{q^2}{8} B_2^{-1} \left[\sin\left(2\psi_s + 2\psi_0\right) \right] ,$$

$$M_2 = -\left(a + b_1\right)^{-1} \left\{ A_1 v_1 + A_{11} \sin v' + A_{12} \sin 2v' \right.$$

$$+ c^2 \left(-2\alpha_1\right)^{\frac{1}{2}} \left(\alpha_2^2 - \alpha_3^2\right)^{-\frac{1}{2}} \eta_0^3 \left[B_1 \psi_1 - \frac{1}{2} \psi_1 \cos\left(2\psi_s + 2\psi_0\right) \right.$$

$$- \frac{q^2}{8} \sin\left(2\,\psi_s + 2\,\psi_0\right) + \frac{q^2}{64} \sin\left(4\,\psi_s + 4\,\psi_0\right)\Big]\Big\} ,$$

$$E_2 = \left[1 - e' \cos\left(\mathcal{E} + E_1\right)\right]^{-1} M_2 ,$$

$$E = \left(\mathcal{E} + E_1 + E_2\right) ,$$

$$\cos v''' = \left[\cos\left(\mathcal{E} + E_1 + E_2\right) - e\right]\left[1 - e \cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1} ,$$

$$\sin v''' = \left(1 - e^2\right)^{\frac{1}{2}} \left[1 - e \cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1} \sin\left(\mathcal{E} + E_1 + E_2\right) ,$$

also, determine v''' within the limits $0 \leq v''' < 2\pi$,

$$v_2 = v''' - \left(v_0 + M_s + v_1\right) = \left(v''' - v''\right) ,$$

$$\psi_2 = \left(-2\alpha_1\right)^{-\frac{1}{2}} \left(\alpha_2^2 - \alpha_3^2\right)^{\frac{1}{2}} \eta_0^{-1} B_2^{-1} \Big\{ A_2\, v_2 + A_{21}\, v_1 \cos v' + 2\, A_{22}\, v_1 \cos 2\, v'$$

$$+ A_{23} \sin 3\, v' + A_{24} \sin 4\, v' \Big\} + \frac{q^2}{4} B_2^{-1} \Big\{ \psi_1 \cos\left(2\,\psi_s + 2\,\psi_0\right)$$

$$+ \frac{3}{8} q^2 \sin\left(2\psi_s + 2\,\psi_0\right) - \frac{3q^2}{64} \sin\left(4\,\psi_s + 4\,\psi_0\right) \Big\} ,$$

$$v = M_s + v_0 + v_1 + v_2 ,$$

$$\psi = \psi_s + \psi_0 + \psi_1 + \psi_2 ,$$

put v and ψ within 2π and into proper quadrant,

$$\sin \chi = \frac{\sqrt{1 - \eta_0^2}\ \sin \psi}{\sqrt{1 - \eta_0^2 \sin^2 \psi}} ,$$

$$\cos \chi = \frac{\cos \psi}{\sqrt{1 - \eta_0^2 \sin^2 \psi}} ,$$

also, determine χ within the limits $0 \leq \chi < 2\pi$,

$$\rho = (1 + e \cos v)^{-1} p ,$$

$$\eta = \eta_0 \sin \psi ,$$

$$\phi = \beta_3 + a_3 \left(a_2^2 - a_3^2\right)^{-\frac{1}{2}} \eta_0 \left\{ \left(1 - \eta_0^2\right)^{-\frac{1}{2}} \left(1 - \eta_2^{-2}\right)^{-\frac{1}{2}} \chi + B_3 \psi + \frac{3}{32} \eta_0^2 \eta_2^{-4} \sin 2\psi \right\} - c^2 a_3 \left(-2a_1\right)^{-\frac{1}{2}} \left\{ A_3 v + \sum_{n=1}^{4} A_{3n} \sin n v \right\} ,$$

the parameters ρ and η should satisfy the following conditions

$$0 \leq \rho < \infty$$

$$-1 \leqq \eta \leqq +1 ,$$

also put ϕ within 2π and into its proper quadrant,

$$h_\rho^2 = \frac{\rho^2 + \eta^2 c^2}{\rho^2 + c^2} ,$$

$$h_\eta^2 = \frac{\rho^2 + \eta^2 c^2}{1 - \eta^2} ,$$

$$h_\phi^2 = \left(\rho^2 + c^2\right)\left(1 - \eta^2\right) ,$$

$$\dot{\rho} = \frac{ae\sqrt{-2\alpha_1\left(\rho^2 + A\rho + B\right)}}{h_\rho^2\left(\rho^2 + c^2\right)} \sin E ,$$

$$\dot{\eta} = \frac{c\eta_0\sqrt{-2\alpha_1\left(\eta_2^2 - \eta^2\right)}}{h_\eta^2\left(1 - \eta^2\right)} \cos \psi ,$$

$$X = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \cos \phi ,$$

$$Y = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \sin \phi ,$$

$$Z = \rho\eta ,$$

$$\dot{X} = X\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) - \frac{Y \cdot \alpha_3}{h_\phi^2} ,$$

$$\dot{Y} = Y\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) + \frac{X \cdot \alpha_3}{h_\phi^2} ,$$

and

$$\dot{Z} = \rho\dot{\eta} + \eta\dot{\rho} .$$

REMARKS

The programming of this procedure is presently being handled by the IBM Space Systems Center, Bethesda, Maryland. On the matter of improvement programs, it should be said that the author is presently working on an orbit correction routine, unique to this system, in that mean-corrected values of the total set of Jacobi-constants are to be obtained for the epoch of an arc of observations and used for prediction purposes.

It is intended that this program will include work being done on the effects of the residual fourth harmonic, the odd harmonics, the tesseral harmonics, lunar-solar forces, and aerodynamic and electromagnetic drag. In this respect, it can be stated that even though it is now possible to do the gravitational theory of a satellite orbit very accurately without use of perturbation theory, Izsak (Reference 4) has stated that the oblateness perturbations unaccounted for by Vinti's potential can be treated by a first order method, that is, without multiplications of Fourier series.

This method shows great promise from the standpoint of computer operations in that it requires a relatively small number of storage locations throughout the entire computing procedure.

Presently, this procedure is being tested with two well-behaved and accurate, high earth satellite orbits: Gamma Ray Astronomy Satellite (1961ν) and Vanguard I (1958β). Preliminary tests on the IBM 7090 indicate that the orbit generator can compute approximately 1440 minute points (time, x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$ each minute for 1440 minutes) in 54 seconds of computer operation with simultaneous production of BCD tape output.

ACKNOWLEDGMENTS

The author is especially grateful to Dr. John P. Vinti of the National Bureau of Standards for his valuable advice in preparing this report, to Mr. Imre G. Izsak of the Smithsonian Institution, Astrophysical Observatory, Cambridge, Massachusetts, and Dr. Peter Musen of the Theoretical Division of NASA's Goddard Space Flight Center for several technical discussions on the subject matter of this report.

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